

A CHEVALLEY'S THEOREM IN CLASS \mathcal{C}^r

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ABSTRACT. Let W be a finite reflection group acting orthogonally on \mathbf{R}^n , P be the Chevalley polynomial mapping determined by an integrity basis of the algebra of W -invariant polynomials, and h be the highest degree of the coordinate polynomials in P . There exists a linear mapping: $\mathcal{C}^r(\mathbf{R}^n)^W \ni f \mapsto F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$ such that $f = F \circ P$, continuous for the natural Fréchet topologies. A general counterexample shows that this result is the best possible. The proof uses techniques of division by linear forms and a study of compensation phenomena. An extension to $P^{-1}(\mathbf{R}^n)$ of invariant formally holomorphic regular fields is needed.

1. INTRODUCTION

Let W be a finite subgroup of $O(n)$ generated by reflections. The algebra of W -invariant polynomials is generated by n algebraically independent W -invariant homogeneous polynomials and the degrees of these basic invariants are uniquely determined. This theorem and its converse were first stated by Shephard and Todd in [20] where the direct statement was proved case by case. Soon after, in [6] Chevalley gave a beautiful unique proof of this direct statement, which is often called 'Chevalley's theorem'.

A W -invariant complex analytic function may be written as a complex analytic function of the basic invariants ([21]). Glaeser's theorem ([11]) shows that real W -invariant functions of class \mathcal{C}^∞ , may be expressed as \mathcal{C}^∞ functions of the basic invariants.

In finite class of differentiability, Newton's theorem in class \mathcal{C}^r ([2]) dealt with symmetric functions and as a consequence with the Weyl group of A_n . This particular case shows a loss of differentiability as already did Whitney's even function theorem ([22]) which established the result for the Weyl group of A_1 . A first attempt to study the general case may be found in the first part of [4] where the best result was obtained for the Weyl groups of A_n, B_n and the dihedral groups $I_2(k)$ by a method which was on the right track but needed an additional ingredient to deal with the general case. The behavior of the partial derivatives of the functions of the basic invariants on the critical image was studied in [12] for A_n, B_n, D_n and $I_2(k)$.

Here we give for any reflection group a result which is the best possible as shown by a general counter example. Let p_1, \dots, p_n be an integrity basis, we define the

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‘Chevalley’ mapping $P : \mathbf{R}^n \ni x \mapsto P(x) = (p_1(x), \dots, p_n(x)) \in \mathbf{R}^n$. The loss of differentiability is governed by the highest degree of the basic invariant polynomials. More precisely we have:

Theorem 1.1. *Let W be a finite group generated by reflections acting orthogonally on \mathbf{R}^n and let f be a W -invariant function of class \mathcal{C}^r on \mathbf{R}^n . There exists a function F of class $\mathcal{C}^{\lfloor r/h \rfloor}$ on \mathbf{R}^n such that $f = F \circ P$, where P is a Chevalley polynomial mapping associated with W and h is the highest degree of the coordinate polynomials in P , equal to the greatest Coxeter number of the irreducible components of W .*

Since a change of basic invariants is an invertible polynomial map on the target, the statement does not depend on the choice of the set of basic invariants.

2. THE CHEVALLEY MAPPING

A detailed study may be found in [5] or [8].

When W is reducible, it is a direct product of its irreducible components, say $W = W^0 \times W^1 \times \dots \times W^s$ and we may write \mathbf{R}^n as an orthogonal direct sum $\mathbf{R}^{n_0} \oplus \mathbf{R}^{n_1} \oplus \dots \oplus \mathbf{R}^{n_s}$ where W^0 is the identity on \mathbf{R}^{n_0} , subspace of W -invariant vectors, and for $i = 1, \dots, s$, W^i is an irreducible finite Coxeter group acting on \mathbf{R}^{n_i} . We may choose coordinates that fit with this orthogonal direct sum. If $w = w_1 \dots w_s \in W$ with $w_i \in W^i$, $1 \leq i \leq s$ we have $w(x) = w(x^0, x^1, \dots, x^s) = (x^0, w_1(x^1), \dots, w_s(x^s))$ for all $x \in \mathbf{R}^n$. The direct product of the identity on \mathbf{R}^{n_0} and Chevalley mappings P^i associated with W^i acting on \mathbf{R}^{n_i} , $1 \leq i \leq s$, is a Chevalley map $P = Id_0 \times P^1 \times \dots \times P^s$ associated with the action of W on \mathbf{R}^n .

For an irreducible W (or for an irreducible component) we will assume as we may that the degrees of the coordinate polynomials p_1, \dots, p_n are in increasing order: $2 = k_1 \leq \dots \leq k_n = h$, Coxeter number of W . In the reducible case we will denote by h the highest degree of the coordinate polynomials which is the maximal Coxeter number of the irreducible components.

Let \mathcal{R} be the set of reflections different from identity in W . The number of these reflections is $\mathcal{R}^\# = d = \sum_{i=1}^n (k_i - 1)$. For each $\tau \in \mathcal{R}$, let λ_τ be a linear form the kernel of which is the hyperplane $H_\tau = \{x \in \mathbf{R}^n | \tau(x) = x\}$. The jacobian of P is $J_P = c \prod_{\tau \in \mathcal{R}} \lambda_\tau$ for some constant $c \neq 0$. The critical set is the union of the H_τ when τ runs through \mathcal{R} .

A Weyl Chamber C is a connected component of the regular set. The other connected components are obtained by the action of W and the regular set is $\bigcup_{w \in W} w(C)$. There is a stratification of \mathbf{R}^n by the regular set, the reflecting hyperplanes H_τ and their intersections. The mapping P induces an analytic diffeomorphism of C onto the interior of $P(\mathbf{R}^n)$ and an homeomorphism that carries the stratification from the fundamental domain \overline{C} onto $P(\mathbf{R}^n)$.

The Chevalley mapping is neither injective nor surjective. Actually the fiber over each point of the image is a W -orbit. The mapping P is proper and separates the orbits ([19]). It is the restriction to \mathbf{R}^n of a complex W -invariant mapping from \mathbf{C}^n onto ([13]) \mathbf{C}^n , still denoted by P .

On its regular set, the complex P is a local analytic isomorphism. Its critical set, where the jacobian vanishes, is the union of the complex hyperplanes $H_\tau = \{z \in \mathbf{C}^n | \tau(z) = z\}$, kernels of the complex forms λ_τ . The critical image is the algebraic

set $\{u \in \mathbf{C}^n \mid \Delta(u) = J_P^2(z) = 0\}$, on which P carries the stratification. The complex P is proper, it is a $W^\#$ -fold covering of \mathbf{C}^n ramified over the critical image.

Finally, there are only finitely many types of irreducible finite Coxeter groups defined by their connected graph types. Even when these groups are Weyl groups of root systems or Lie algebras, we will follow the general usage and denote them with upper case letters.

3. WHITNEY FUNCTIONS AND r -REGULAR, m -CONTINUOUS JETS.

A complete study of Whitney functions may be found in [21].

A jet of order $m \in \mathbf{N}$, on a locally closed set $E \subset \mathbf{R}^n$ is a collection $A = (a_k)_{\substack{k \in \mathbf{N}^n \\ |k| \leq m}}$ of real valued functions a_k continuous on E . At each point $x \in E$ the jet A determines a polynomial $A_x(X)$, and we sometimes speak of continuous polynomial fields instead of jets ([15]). As a function, A_x acts upon vectors $x' - x$ tangent to \mathbf{R}^n at x . To avoid introducing the notation $T_x^r A$, we write somewhat inconsistently:

$$A_x : x' \mapsto A_x(x') = \sum_k \frac{1}{k!} a_k(x) (x' - x)^k.$$

By formal derivation of A of order $q \in \mathbf{N}^n$, $|q| \leq m$, we get jets of the form $(a_{q+k})_{|k| \leq m-|q|}$ inducing polynomials

$$(D^q A)_x(x') = \left(\frac{\partial^{|q|} A}{\partial x^q} \right)_x (x') = a_q(x) + \sum_{\substack{k \geq q \\ |k| \leq m}} \frac{1}{(k-q)!} a_k(x) (x' - x)^{k-q}.$$

For $0 \leq |q| \leq r \leq m$, we put:

$$(R_x A)^q(x') = (D^q A)_{x'}(x') - (D^q A)_x(x').$$

Definition 3.1. Let A be an m -jet on E . For $r \leq m$, A is r -regular on E , if and only if for all compact set K in E , for $(x, x') \in K^2$, and for all $q \in \mathbf{N}^n$ with $|q| \leq r$, it satisfies the Whitney conditions.

$$(\mathcal{W}_q^r) \quad (R_x A)^q(x') = o(|x' - x|^{r-|q|}), \text{ when } |x - x'| \rightarrow 0.$$

Remark 3.2. Even if $m > r$ there is no need to consider the truncated field A^r instead of A in the conditions (\mathcal{W}_q^r) . Actually $(R_x A^r)^q(x')$ and $(R_x A)^q(x')$ differ by a sum of terms $[a_k(x)/(k-q)!] (x' - x)^{k-q}$, with a_k uniformly continuous on K and $|k| - |q| > r - |q|$.

The space of r -regular jets of order m on E , is naturally provided with the Fréchet topology defined by the family of semi-norms:

$$\|A\|_{K_n}^{r,m} = \sup_{\substack{x \in K_n \\ |k| \leq m}} \left| \frac{1}{k!} a_k(x) \right| + \sup_{\substack{(x, x') \in K_n^2 \\ x \neq x', |k| \leq r}} \left(\frac{|(R_x A)^k(x')|}{|x - x'|^{r-|k|}} \right)$$

where K_n runs through a countable exhaustive collection of compact sets of E . Provided with this topology the space of r -regular, m -continuous polynomial fields on E is a Fréchet space denoted by $\mathcal{E}^{r,m}(E)$. If $r = m$, $\mathcal{E}^r(E)$ is the space of Whitney fields of order r or Whitney functions of class C^r on E .

Theorem 3.3. Whitney extension theorem ([23]). *The restriction mapping of the space $\mathcal{E}^r(\mathbf{R}^n)$ of functions of class C^r on \mathbf{R}^n to the space $\mathcal{E}^r(E)$ of Whitney fields of order r on E , is surjective. There is a linear section, continuous when the spaces are provided with their natural Fréchet topologies.*

Let E be a closed subset of $\mathbf{C}^n \simeq \mathbf{R}^{2n}$, we consider jets A on E with complex valued coefficients a_k . They induce in $z \in E$ the polynomials:

$$A_z(X, Y) = \sum_{|k|+|l| \leq m} \frac{1}{k!l!} a_{k,l}(z) X^k Y^l \in \mathbf{C}[X, Y],$$

and we may define the Fréchet space $\mathcal{E}^r(E; \mathbf{C})$ of complex valued Whitney functions of class \mathcal{C}^r .

Definition 3.4. ([15]) A Whitney function $A \in \mathcal{E}^r(E; \mathbf{C})$ is formally holomorphic if it satisfies the Cauchy-Riemann equalities:

$$i \frac{\partial A}{\partial X_j} = \frac{\partial A}{\partial Y_j}, \quad j = 1, \dots, n.$$

Let $Z = (Z_1, \dots, Z_n)$, $Z_j = X_j + iY_j$, $j = 1, \dots, n$. The field A is formally holomorphic if and only if $\partial A / \partial \overline{Z_j} = 0$, $j = 1, \dots, n$. Thus for all $z \in E$ the polynomial A_z belongs to $\mathbf{C}[Z]$ and is of the form $A_z(Z) = \sum_k \frac{1}{k!} a_k(z) Z^k$.

The algebra of formally holomorphic Whitney functions of class \mathcal{C}^r on the (locally) closed set E of \mathbf{C}^n will be denoted by $\mathcal{H}^r(E)$. It is a closed sub-algebra of $\mathcal{E}^r(E; \mathbf{C})$ and therefore a Fréchet space when provided with the induced topology. In practice we define the semi-norms $\|A\|_r^{K_n}$ on $\mathcal{H}^r(E)$ by the same formulas as in $\mathcal{E}^r(E; \mathbf{R})$, only using modulus instead of absolute value.

To take advantage of compensation phenomenons, it may be convenient to consider Fréchet spaces $\mathcal{H}^{r,m}(E)$ of formally holomorphic r -regular jets of order $m \geq r$ on E .

Definition 3.5. A real form ([17]) or real situated subspace ([15]) of \mathbf{C}^n is a real vector subspace E of real dimension n such that $E \oplus iE = \mathbf{C}^n$.

A real form is a real subspace $E_S = \{z \in \mathbf{C}^n | Sz = z\}$, where S is an anti-involution.

Example. Let α be an involution of $\{1, \dots, n\}$, $\Gamma_\alpha = \{z \in \mathbf{C}^n | z_{\alpha(j)} = \overline{z_j}, j = 1, \dots, n\}$ is a real form of \mathbf{C}^n defined by the anti-involution $z \mapsto \alpha(z)$.

Let W be a finite reflection group acting orthogonally on \mathbf{R}^n and P be its Chevalley polynomial mapping as above. Since P is defined over \mathbf{R} (its coefficients are real), $P^{-1}(\mathbf{R}^n)$ is the union of real forms $\Gamma_{S_w} \subset \mathbf{C}^n$, where w runs through the involutions of W and S_w is the anti-involution defined by $S_w(u + iv) = wu - i wv$.

Let $f \in \mathcal{C}^r(\mathbf{R}^n)^W$ be a W -invariant function of class \mathcal{C}^r . It induces on \mathbf{R}^n a W -invariant Whitney field of order r and a formally holomorphic field in $\mathcal{H}^r(\mathbf{R}^n)^W$ which will still be denoted by f . By using Whitney's extension theorem, one may show ([2]) that there is a linear and continuous extension ¹:

$$\mathcal{H}^r(\mathbf{R}^n)^W \ni f \mapsto \tilde{f} \in \mathcal{H}^r(P^{-1}(\mathbf{R}^n))^W.$$

¹This extension will allow us to get a field \tilde{F} on \mathbf{R}^n with $\tilde{f} = \tilde{F} \circ P$ and derive its regularity from its continuity by using the below lemma 4.4. This process might be avoided if there was an available proof of the Whitney regularity property ([24]) of $P(\mathbf{R}^n)$, a most likely conjecture proved for A_n in [14], easy to show for some lower dimensional reflection groups (e.g. $I_2(k), H_3$). Unfortunately there is none, although in [9] a key point for the proof of this conjecture is studied and a sketchy proof of it is given for several Coxeter groups.

4. SOME MULTIPLICATION AND DIVISION PROPERTIES.

Lemma 4.1. *Let Γ be a finite union of real forms of \mathbf{C}^n , A be in $\mathcal{H}^r(\Gamma)$, and Q be a polynomial $(s-1)$ -flat on S . Let $z \in \Gamma$ and $z_0 \in S \cap \Gamma$, then for all $q \in \mathbf{N}^n, |q| \leq r$:*

$$(R_{z_0}QA)^q(z) = (D^qQA)_z(z) - (D^qQA)_{z_0}(z) \in o(|z - z_0|^{r-|q|+s}).$$

Moreover $QA \in \mathcal{H}^{r+s}(S \cap \Gamma)$ and is $(s-1)$ -flat on $S \cap \Gamma$. For all compact $K \subset S \cap \Gamma$, there exists a numerical constant c such that $\|QA\|_K^{r+s} \leq c\|Q\|_K^{r+s}\|A\|_K^r$.

Proof. Let $z_0 \in S \cap \Gamma$. For all $z \in \Gamma$, all $q \in \mathbf{N}^n, |q| \leq r$, and $p \leq q$, we consider:

$$(D^pQ)_z(z)(D^{q-p}A)_z(z) - (D^pQ)_{z_0}(z)(D^{q-p}A)_{z_0}(z).$$

By Taylor's formula for polynomials $(D^pQ)_z(z) = (D^pQ)_{z_0}(z)$, and this difference is:

$$(D^pQ)_z(z) [(D^{q-p}A)_z(z) - (D^{q-p}A)_{z_0}(z)].$$

By assumption $(D^{q-p}A)_z(z) - (D^{q-p}A)_{z_0}(z) \in o(|z - z_0|^{r-|q|+|p|})$, and for $|p| < s$ $(D^pQ)_z(z) \in O(|z - z_0|^{s-|p|})$. The product is in $o(|z - z_0|^{r-|q|+s})$ either because $|p| < s$ and $r - |q| + |p| + s - |p| = r - |q| + s$ or because $|p| \geq s$ and $r - |q| + |p| \geq r - |q| + s$.

The behavior of $(R_{z_0}QA)^q(z)$ is now a consequence of Leibniz' derivation formula.

Actually $QA \in \mathcal{H}^{r,r+s}$. On $S \cap \Gamma$, $|p| < s \Rightarrow (D^pQ)_{z_0}(z_0) = 0$, therefore in the derivatives of QA of order $\leq r+s$ the only derivatives of A that are not multiplied by a derivative of Q that vanishes, are of order $\leq r$. Then the above estimates show that when $|q| \leq r+s$, the field QA satisfies Whitney conditions \mathcal{W}_q^{r+s} on $S \cap \Gamma$. Thus $QA \in \mathcal{H}^{r+s}(S \cap \Gamma)$, and clearly it is $(s-1)$ -flat on $S \cap \Gamma$.

This situation was already noticed in [10]: when multiplying a field r_1 -regular, (s_1-1) -flat by a field r_2 -regular, (s_2-1) -flat, the product is $\min(r_1+s_2, r_2+s_1)$ -regular and (s_1+s_2-1) flat. Here, on $S \cap \Gamma$, we have $r_1 = r$, $s_1 = 0$ for A and $r_2 = +\infty$, $s_2 = s$ for Q . \square

Example. Let Q be an homogeneous polynomial of degree s . It vanishes at the origin with all its derivatives of order $\leq s-1$. If $A \in \mathcal{H}^r(\Gamma)$, for all $z \in \Gamma$ and all $q \in \mathbf{N}^n, |q| \leq r$:

$$(R_0QA)^q(z) = (D^qQA)_z(z) - (D^qQA)_0(z) \in o(|z|^{r+s-|q|}).$$

The same result holds if instead of a product QA we have a sum $\sum_{i=1}^n Q_i A_i$, with homogeneous polynomials Q_i of degree $s_i \geq s$ and $A_i \in \mathcal{H}^r(\Gamma)$.

Let us recall the following division lemma:

Lemma 4.2 ([2]). *Let Γ be a finite union of real forms of \mathbf{C}^n , and $\lambda \neq 0$ be a complex linear form with kernel H . If $A \in \mathcal{H}^r(\Gamma)$ is such that $A_z(Z)$ is divisible by $\lambda_z(Z)$ whenever $z \in \Gamma \cap H$ then there exists a field $B \in \mathcal{H}^{r-1}(\Gamma)$ such that $A^r = (\lambda B)^r$. For all compact $K \subset \Gamma$, there exists a constant c such that $\|B\|_K^{r-1} \leq c\|A\|_K^r$.*

Actually $B \in \mathcal{H}^r(\Gamma \setminus H)$ and if $|s| = r$, then $\lambda(z)(D^s B)_z(z)$ tends to zero with $\lambda(z)$.

Remark 4.3. The lemma still holds if we replace Γ by its intersection with one or several hyperplanes distinct from H .

The proof of lemma 4.2 relies upon a consequence of the mean value theorem that will be instrumental in what follows:

Lemma 4.4 ([1],[15]). *Let Γ be a finite union of real forms of \mathbf{C}^n , $\Delta \neq 0$ be a polynomial, and $X = \{x \in \mathbf{C}^n \mid \Delta(x) = 0\}$. If $f \in \mathcal{H}^r(\Gamma \setminus X)$ is r -continuous on Γ , then $f \in \mathcal{H}^r(\Gamma)$.*

Let $(\lambda_\tau)_{\tau \in \mathcal{D}}$ be $\mathcal{D}^\# = d$ non zero complex linear forms with kernels $(H_\tau)_{\tau \in \mathcal{D}}$ all distinct. The hyperplanes $(H_\tau)_{\tau \in \mathcal{D}}$ and their intersections induce a stratification on Γ . Let S_p be a stratum, connected component of the intersection of Γ and exactly p of these hyperplanes, say $(H_\tau)_{\tau \in \mathcal{B}_p}$, $\mathcal{B}_p^\# = p$. The border $\overline{S_p} \setminus S_p$ is a union $\bigcup S_{p+l}$ of strata of lower dimensions, containing $S_d = \Gamma \cap (\bigcap_{\tau \in \mathcal{D}} H_\tau)$. Using these notations we have:

Lemma 4.5. *For $i = 1, \dots, n$, let A_i be in $\mathcal{H}^r(\Gamma)$ and Q_i be an homogeneous polynomial $(s_p - 1)$ -flat on S_p and more generally $(s_{p+l} - 1)$ -flat on each of the S_{p+l} . Assume $p + l - s_{p+l}$ is an increasing function of l and that $A = \sum_{i=1}^n Q_i A_i = (\prod_{\tau \in \mathcal{D}} \lambda_\tau) C$, meaning that:*

$$\forall \mathcal{U} \subseteq \mathcal{D}, \quad A_z(Z) \text{ is divisible by } \prod_{\tau \in \mathcal{U}} \lambda_\tau(Z) \text{ when } z \in \Gamma \cap \left(\bigcap_{\tau \in \mathcal{U}} H_\tau \right).$$

The field C is in $\mathcal{H}^{r+s_p-p}(S_p)$ and its coefficients of order $\leq r + s_d - d$ are continuous on $\overline{S_p}$.²

Proof. By lemma 4.1, $\sum_{i=1}^n Q_i A_i$ is in $\mathcal{H}^{r+s_p}(S_p)$ and in $\mathcal{H}^{r+s_{p+l}}(S_{p+l})$. By lemma 4.2, the field C is in $\mathcal{H}^{r+s_p-p}(S_p)$ and in $\mathcal{H}^{r+s_{p+l}-(p+l)}(S_{p+l})$. We are just to show the continuity on $\overline{S_p}$ of the coefficients of order $\leq r + s_d - d$ in C .

Let S_{p+q} be one of the strata of largest dimension in $\overline{S_p} \setminus S_p$, and let \mathcal{B}_q with $\mathcal{B}_q^\# = q$, be the subset of \mathcal{D} such that S_{p+q} is a connected component of the intersection of Γ and the hyperplanes $(H_\tau)_{\tau \in \mathcal{B}_p \cup \mathcal{B}_q}$, but no other. We may have $q = 1$ but not necessarily since the addition of one hyperplane may automatically entail the addition of some more.

We put: $A = (\prod_{\tau \in \mathcal{B}_p} \lambda_\tau) (\prod_{\tau \in \mathcal{B}_q} \lambda_\tau) (\prod_{\tau \in \mathcal{D} \setminus (\mathcal{B}_p \cup \mathcal{B}_q)} \lambda_\tau) C$ and define:

$$C^1 = (\prod_{\tau \in \mathcal{D} \setminus (\mathcal{B}_p \cup \mathcal{B}_q)} \lambda_\tau) C, \quad \text{and} \quad B = (\prod_{\tau \in \mathcal{B}_q} \lambda_\tau) C^1.$$

On S_p , B is in \mathcal{H}^{r+s_p-p} and so are C and C^1 . On S_{p+q} , C and C^1 are in $\mathcal{H}^{r+s_{p+q}-(p+q)}$.

Let z_0 be the orthogonal projection on S_{p+q} of some $z \in \Gamma$. By lemma 4.1, we have:

$$A_z(z) - A_{z_0}(z) = \left[\prod_{\tau \in \mathcal{B}_p} \lambda_\tau(z) \right] [B_z(z) - B_{z_0}(z)] \in o(|z - z_0|^{r+s_{p+q}}).$$

Let π be a derivation of order p , by Leibniz derivation formula:

$$\begin{aligned} D^\pi A_z(z) - D^\pi A_{z_0}(z) &= \left[\prod_{\tau \in \mathcal{B}_p} \lambda_\tau(z) \right] [D^\pi B_z(z) - D^\pi B_{z_0}(z)] + \dots \\ &\dots + c [B_z(z) - B_{z_0}(z)] \in o(|z - z_0|^{r+s_{p+q}-p}) \end{aligned}$$

for some constant $c \neq 0$.

²Actually since $\overline{S_p}$ is convex and thus Whitney 1-regular ([24]), lemma 4.5 yields that C is $(r + s_d - d)$ -regular on $\overline{S_p}$.

For the remaining part of the proof, we assume that z is in S_p . Then for all $\tau \in \mathcal{B}_p$, $\lambda_\tau(z) = 0$ and we get:

$$B_z(z) - B_{z_0}(z) \in o(|z - z_0|^{r+s_{p+q}-p}).$$

More generally, by considering derivations of order $\pi + \kappa$, with $|\kappa| = k \leq r + s_p - p$ we would get in the same way:

$$D^\kappa B_z(z) - D^\kappa B_{z_0}(z) \in o(|z - z_0|^{r+s_{p+q}-p-k}).$$

Let us put $E = \Gamma \cap (\bigcap_{\tau \in \mathcal{B}_p} H_\tau)$ and $F = H_\tau$ for some $\tau \in \mathcal{B}_q$. Since E and F are \mathbf{R} -linear subspaces, there exists a constant $a > 0$ such that for all x , $d(x, E) + d(x, F) \geq a d(x, E \cap F)$ or equivalently, a constant $b > 0$ such that for all $x \in E$, $d(x, F) \geq b d(x, E \cap F)$. We say that E and F are regularly separated.³ The regular separation brings the existence of constants c_τ such that:

$$\forall \tau \in \mathcal{B}_q, \quad |z - z_0| \leq c_\tau d(z, H_\tau) = c_\tau |\lambda_\tau(z)|.$$

Therefore, since

$$B_z(z) - B_{z_0}(z) = \left[\prod_{\tau \in \mathcal{B}_q} \lambda_\tau(z) \right] [C_z^1(z) - C_{z_0}^1(z)] \in o(|z - z_0|^{r+s_{p+q}-p}),$$

we get that $C_z^1(z) - C_{z_0}^1(z) \in o(|z - z_0|^{r+s_{p+q}-(p+q)})$.

Let us assume by induction that for $|l| \leq k-1 < r + s_{p+q} - (p+q)$:

$$D^l C_z^1(z) - D^l C_{z_0}^1(z) \in o(|z - z_0|^{r+s_{p+q}-(p+q)-|l|}).$$

By Leibniz derivation formula, for j , $|j| = k$:

$$\begin{aligned} D^j B_z(z) - D^j B_{z_0}(z) &= \left[\prod_{\tau \in \mathcal{B}_q} \lambda_\tau(z) \right] (D^j C_z^1(z) - D^j C_{z_0}^1(z)) + \\ &+ \sum_{k-q \leq |j_i| = k-l \leq k-1} a_{q-l} \left[\prod_{q-l} \lambda_\tau(z) \right] (D^{j_i} C_z^1(z) - D^{j_i} C_{z_0}^1(z)) \in o(|z - z_0|^{r+s_{p+q}-p-k}). \end{aligned}$$

The $\left[\prod_{q-l} \lambda_\tau(z) \right]$ stand for $D^{j-j_i}(\prod_{\tau \in \mathcal{B}_q} \lambda_\tau)(z)$, up to a constant factor included in a_{q-l} . Applying the induction assumption to the derivations D^{j_i} of order $\leq k-1$, we see that each term of the sum is in

$$o(|z - z_0|^{r+s_{p+q}-(p+q)-(k-l)+q-l}) = o(|z - z_0|^{r+s_{p+q}-p-k})$$

and thus, that the first term also is. Then, using the regular separation as above, we obtain:

$$D^j C_z^1(z) - D^j C_{z_0}^1(z) \in o(|z - z_0|^{r+s_{p+q}-(p+q)-k}).$$

This completes the induction, and shows that the coefficients of C^1 of order $\leq r + s_{p+q} - (p+q)$ are continuous in z_0 . Since they are continuous on S_{p+q} , by using the triangular inequality, we get their continuity on $S_p \cup S_{p+q}$.

Any $z_1 \in S_{p+q}$ has a neighborhood which does not meet any of the H_τ but those containing z_1 and in this neighborhood, the continuity of the coefficients of C and C^1 are the same. Hence the continuity on $S_p \cup S_{p+q}$ of the coefficients of C of order $\leq r + s_{p+q} - (p+q)$.

³This is a trivial case of regular separation and does not need a general study. The interested reader may take a look at [16] or [21]. The regular separation of real forms was implicitly used in the above extension of f to $P^{-1}(\mathbf{R}^n)$

We can get an analogous result for each stratum $S_{p+q'}$ of maximal dimension in $\overline{S_p} \setminus S_p$ (q and q' may be equal but \mathcal{B}_q and $\mathcal{B}_{q'}$ are distinct).

We then proceed with the strata of highest dimension in $\overline{S_{p+q}} - S_{p+q}$ to get the continuity on $S_{p+q} \cup S_{(p+q)+l}$ of the coefficients of order at most $r + s_{p+q+l} - (p+q+l)$. The continuity on $S_p \cup S_{p+q} \cup S_{(p+q)+l}$ when $z \in S_p$ tends to $z_1 \in S_{(p+q)+l}$ is obtained by the triangular inequality, considering the orthogonal projection z_0 of z on S_{p+q} . We can do the same for each of the $\overline{S_{p+q'}} - S_{p+q'}$, and continue the processes until we reach S_d through all possible paths, getting the continuity of coefficients of order at most $r + s_d - d$. Thus we have the continuity of these coefficients between any two points in S_{p+l} and $S_{(p+l)+m}$.

About the continuity between points in $\overline{S_{p+q}}$ and $\overline{S_{p+q'}}$, if they are in the intersection we already have the continuity, but if they are not, we consider their orthogonal projections on the intersection and thanks to the regular separation of the strata, we can get the continuity by the triangular inequality.

Then the global $(r + s_d - d)$ -continuity on $\overline{S_p}$ is clear. \square

Remark 4.6. When $p = 0$, the strata of type S_0 are open and $s_0 = 0$. For $q = 1$, the first step is given by lemma 4.2.

5. PROOF OF THEOREM 1.1.

We consider an invariant function $f \in \mathcal{C}^r(\mathbf{R}^n)^W$. This function or rather the formally holomorphic field it induces on \mathbf{R}^n has a linear and continuous extension $\tilde{f} \in \mathcal{H}^r(P^{-1}(\mathbf{R}^n))^W$.

Pointwise solution.

Lemma 5.1 ([4]). *For all W -invariant, formally holomorphic polynomial field \tilde{f} of degree r on $P^{-1}(\mathbf{R}^n)$, there exists a formally holomorphic polynomial field \tilde{F} of degree at most r such that for all $z \in P^{-1}(\mathbf{R}^n)$, $\tilde{f}_z = (\tilde{F}_{P(z)} \circ P)_z^r$.*

Proof. On the complement of $\Gamma \cap \bigcup_{\tau \in \mathcal{R}} H_\tau$ in Γ , the mapping P is a local analytic isomorphism and this yields the construction of $\tilde{F} = (\tilde{f} \circ P^{-1})^r$, unambiguously since both \tilde{f} and P are W -invariant. On the regular image of P , \tilde{F} verifies $\tilde{f}^r = (\tilde{F} \circ P)^r$.

Let $x \in \Gamma \cap (\bigcup_{\tau \in \mathcal{R}} H_\tau)$ and let W_x be the isotropy subgroup of W at x . The polynomial \tilde{f}_x is W_x -invariant since for all $w_0 \in W_x \subset W$: $\tilde{f}_x(X) = \tilde{f}_{w_0x}(w_0X) = \tilde{f}_x(w_0X)$ where the first equality results from the W -invariance of the field \tilde{f} and the second from $w_0x = x$. As a consequence, \tilde{f}_x is a polynomial in the W_x -invariant generators $v = (v_1, \dots, v_n)$ of the subalgebra of W_x -invariant polynomials, and we have $\tilde{f}_x = Q \circ v$.

There exists a neighborhood of x in \mathbf{C}^n which does not meet any of the hyperplanes H_τ but those containing x . In this neighborhood we may write $P = q \circ v$ for some polynomial q , since P is W_x -invariant. Up to a multiplicative constant the jacobian of q at $v(x)$ is the product $\prod_{\lambda_s(x) \neq 0} \lambda_s$ and q is an analytic isomorphism in a neighborhood of $v(x)$.

We define the jet at $P(x)$ by $\tilde{F}_{P(x)} = [Q \circ q^{-1}]^r$ and get:

$$[\tilde{F} \circ P]_x^r = [(Q \circ q^{-1})^r \circ (q \circ v)]_x^r = [(Q \circ q^{-1}) \circ (q \circ v)]_x^r = (Q \circ v)_x = \tilde{f}_x. \quad \square$$

Remark 5.2. When the isotropy subgroup of x_0 is W itself, $\forall w \in W$, $\tilde{f}_{x_0}(X) = \tilde{f}_{wx_0}(wX) = \tilde{f}_{x_0}(wX)$. This means that $\tilde{f}_{x_0}(X)$ is W -invariant and by the polynomial Chevalley's theorem, that $\tilde{f}_{x_0}(X) = Q_0(P(X))$. The polynomial $Q_0 = \tilde{F}_{P(x_0)}$ of weight r is of degree $[r/h]$ in the invariant polynomial p of highest degree h . The result announced in theorem 1.1 fits with the formal computation.

A criterion of regularity for \tilde{F} . When $\tilde{f} \in \mathcal{H}^r(P^{-1}(\mathbf{R}^n))$, the above proof of lemma 5.1 shows that $\tilde{F} = (\tilde{f} \circ P^{-1})^r$ is r -regular on the complement in \mathbf{R}^n of the critical image $\{u \in \mathbf{C}^n \mid \Delta(u) = 0\}$. The discriminant Δ is a polynomial, therefore by lemma 4.5 it will be sufficient to prove that \tilde{F} is $[r/h]$ -continuous on \mathbf{R}^n to get its $[r/h]$ -regularity.

Since P is proper the continuity of $\tilde{F}_\alpha \circ P$, entails the continuity of the coefficient \tilde{F}_α . So let us check the continuity of the $\tilde{F}_\alpha \circ P$ when $|\alpha| \leq [r/h]$.

Clearly $\tilde{F}_0 \circ P = \tilde{f}_0$ is continuous. For the first derivatives, it is natural to consider the partial derivatives of \tilde{f} , and get the system:

$$(I) \quad \left(\frac{\partial \tilde{f}}{\partial z} \right) = \left(\left(\frac{\partial p_i}{\partial z_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \right) \left(\frac{\partial \tilde{F}}{\partial p} \circ P \right).$$

If we show that the loss of differentiability from $\tilde{f} = \tilde{F} \circ P$ to $(\partial \tilde{F} / \partial p) \circ P$ when solving (I) is of h units, applying the same process to $\tilde{g}_j = (\partial \tilde{F} / \partial p_j) \circ P$ instead of $\tilde{f} = \tilde{F} \circ P$, at the next step again there will be a loss of differentiability of h units. An induction would show that for $|\alpha| \leq [r/h]$, the mappings $(\partial^{|\alpha|} \tilde{F} / \partial p^\alpha) \circ P$ are continuous on $P^{-1}(\mathbf{R}^n)$ and since P is proper, that the derivatives $\partial^{|\alpha|} \tilde{F} / \partial p^\alpha$ of \tilde{F} are continuous on \mathbf{R}^n .

Conclusion: To complete the proof, we just have to show that when solving (I), $\forall j = 1, \dots, n$ we get: $(\partial \tilde{F} / \partial p_j) \circ P \in \mathcal{H}^{r-h}(P^{-1}(\mathbf{R}^n))$.

Solving (I), Reduction to the irreducible case. Using Cramer's method, we multiply both sides by the adjoint matrix of the system. Since the jacobian determinant is $c(\prod_{\tau \in \mathcal{R}} \lambda_\tau)$, we get:

$$(II) \quad \left\{ c \left(\prod_{\tau \in \mathcal{R}} \lambda_\tau \right) \frac{\partial \tilde{F}}{\partial p_j} \circ P = \sum_{i=1}^n (-1)^{i+j} M_{i,j} \frac{\partial \tilde{f}}{\partial z_i}, j = 1, \dots, n \right.$$

From (II), $\forall \tau \in \mathcal{R}$, if $\lambda_\tau(z) = 0$ the polynomial $\left(\sum_{i=1}^n (-1)^{i+j} M_{i,j} (\partial \tilde{f} / \partial z_i) \right)_z (Z)$ is divisible by $\lambda_\tau(Z)$, and since the λ_τ are pairwise relatively prime, $\forall \mathcal{U} \subseteq \mathcal{R}$ if $z \in \bigcap_{\tau \in \mathcal{U}} H_\tau$, then $\left(\sum_{i=1}^n (-1)^{i+j} M_{i,j} (\partial \tilde{f} / \partial z_i) \right)_z (Z)$ is divisible by $\prod_{\tau \in \mathcal{U}} \lambda_\tau(Z)$.

In the reducible case, in convenient bases, the jacobian matrix of P is block diagonal. We have $J_P = J_{P^1} \dots J_{P^s}$ and $M_{i,j}$ is of the form $J_{P^1} \dots J_{P^{l-1}} M_{i,j}^l J_{P^{l+1}} \dots J_{P^s}$ where $M_{i,j}^l$ is a minor of the $n_l \times n_l$ block associated with P^l , Chevalley map of W^l . After simplification we see that it is sufficient to study each block.

For any finite reflection group:

$$h = k_n = \sum_{1 \leq j \leq n} (k_j - 1) - \sum_{1 \leq j \leq n-1} (k_j - 1) + 1,$$

where the first sum $\sum_{1 \leq j \leq n} (k_j - 1)$ is the degree of the jacobian determinant, equal to the number of linear forms λ_τ which is the number $d = \mathcal{R}^\#$ of reflections in W . The second sum $\sum_{1 \leq j \leq n-1} (k_j - 1)$ is the least degree s of the minors $M_{i,j}$ of the jacobian determinant of system (I).

If W is reducible, the formula also holds for each irreducible component. In particular for any component with the greatest Coxeter number h , we have $h = 1 + d' - s'$ where d' and s' are those associated with this irreducible component. Accordingly, we may and will assume from now on, without loss of generality, that W is an irreducible Coxeter group.

Stratification and Compensation by the $M_{i,j}$. We denote by S_p a stratum in $P^{-1}(\mathbf{R}^n)$, which is a connected component of the intersection of Γ and exactly p of the reflecting hyperplanes. The points of each stratum are stabilized by the same isotropy group, subgroup of W generated by reflections about the hyperplanes containing the stratum. The different possible isotropy subgroups and strata types may be determined from the Dynkin diagram. The stratum of dimension 0 is the origin. The strata of dimension 1 are those determined by removing only one point in the Dynkin diagram, they are strata S_q such that their closure is $\overline{S_q} = S_q \cup \{0\}$. At the other end the strata of dimension n are the connected components of the regular set in Γ .

On any stratum S_p , by lemma 4.1 and lemma 4.2, $(\partial \tilde{F} / \partial p_j) \circ P \in \mathcal{H}^{r-1+s_p-p}(S_p)$, if $M_{i,j}$ is at least $(s_p - 1)$ -flat on S_p . In a neighborhood of $z \in S_p$ we have $P = q \circ v$ with q invertible and v the Chevalley mapping of W_z , isotropy subgroup of z (and of any point in S_p). Observe that W_z is not irreducible: it is not essential since it stabilizes S_p , and it may also have several irreducible components. The adjoint of the jacobian matrix of P which is the transpose of its comatrix is the product in this order of the adjoint of the jacobian matrices V and Q of v and q respectively. We have $M_{i,j} = \sum_{k=1}^n V_{k,j} Q_{i,k}$. The $V_{k,j}$ and accordingly the $M_{i,j}$ are $(s_z - 1)$ -flat on S_p . So $s_p = s_z$ and, since p is the number d_z of reflections in W_z , $1 - s_p + p$ is the Coxeter number $h_z = 1 - s_z + d_z$ of W_z .

The isotropy group of the points $z \in S_p$ is a subgroup of the isotropy group of the points $z' \in S_{p+q}$. Therefore $h_z \leq h_{z'}$, or $1 + p - s_p \leq 1 + p + q - s_{p+q}$. Also $1 + d - s_d = h$ is larger than $h_z = 1 + d_z - s_z$ for any $z \neq 0$.

Finally, the minors $M_{i,j}$ are homogeneous polynomials of degree:

$$s_j = \sum_{1 \leq u \leq n, u \neq j} (k_u - 1) \geq s = \sum_{1 \leq u \leq n-1} (k_u - 1).$$

They are at least $(s - 1)$ -flat on the intersection of the $\mathcal{R}^\# = d$ reflecting hyperplanes. (Observe that this might be used to get $s_p = s_z$ by induction).

In (II), lemma 4.5 applies to the closure of each connected component of the regular set, with $A_i = \partial \tilde{f} / \partial z_i \in \mathcal{H}^{r-1}(P^{-1}(\mathbf{R}^n))$, $Q_i = (-1)^{i+j} M_{i,j}$, and gives the $[(r-1) + s - d]$ -continuity of the $(\partial \tilde{F} / \partial p_j) \circ P$ on their union $P^{-1}(\mathbf{R}^n)$. The result we needed to complete the proof:

$$\frac{\partial \tilde{F}}{\partial p_j} \circ P \in \mathcal{H}^{r-1-d+s}(P^{-1}(\mathbf{R}^n)), \quad 1 + d - s = h$$

is now a consequence of lemma 4.4.⁴ \square

Remark 5.3. The above result gives a loss of differentiability of $1 + d - s$, where s is the least degree of the $M_{i,j}$. Actually $M_{i,j}$ is the jacobian of the polynomial mapping:

$$(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n; z_i) \mapsto (p_1(z), \dots, p_{j-1}(z), p_{j+1}(z), \dots, p_n(z); z_i).$$

This mapping is invariant by the subgroup W_i of W that leaves invariant the i^{th} coordinate axis in \mathbf{R}^n , say $\mathbf{R} \mathbf{e}_i$ ([3]). This subgroup W_i is generated by the subset $\mathcal{R}_i \subset \mathcal{R}$ of the reflections it contains. These are the reflections α in W such that $\alpha(\mathbf{e}_i) = \mathbf{e}_i$, about the hyperplanes H_α containing \mathbf{e}_i ⁵. The $M_{i,j}$, $j = 1, \dots, l$, as jacobians of W_i -invariant polynomial mappings are polynomial multiples of $(\prod_{\tau \in \mathcal{R}_i} \lambda_\tau)$. In [4] the formula for the loss of differentiability at each step was also of the form $1 + d - s$, but s was $\min_{1 \leq i \leq n} \mathcal{R}_i^\#$. Clearly $\min_{1 \leq i \leq n} \mathcal{R}_i^\# \leq \min_{1 \leq i, j \leq n} \text{degree} M_{i,j}$. In some cases $(A_n, B_n, I_2(k))$ the equality holds but in general the loss of differentiability given by [4] was not the best possible. Considering H_3 for an example, we now have $s = 6$ instead of 2, and the class of differentiability of F is $[r/10]$ instead of $[r/14]$.

All the operations from $f \in \mathcal{C}^r(\mathbf{R}^n)^W$ up to $F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$ are linear and continuous when using the natural Fréchet topologies. A modulus of continuity for the Whitney conditions could be followed from $\|f\|^r$ to $\|F\|^{[r/h]}$. So Chevalley's theorem in class \mathcal{C}^r may be restated as:

Theorem 1.1. *Let W be a finite group generated by reflections acting orthogonally on \mathbf{R}^n , P the Chevalley polynomial mapping associated with W , and $h = k_n$ the highest degree of the coordinate polynomials in P (equal to the greatest Coxeter number of the irreducible components of W). There exists a linear and continuous mapping:*

$$\mathcal{C}^r(\mathbf{R}^n)^W \ni f \rightarrow F \in \mathcal{C}^{[r/h]}(\mathbf{R}^n)$$

such that $f = F \circ P$.

Remark 5.4. Theorem 1.1 gives global results. As it is clear from the proof, in the neighborhood of each point x the loss of differentiability is governed by the isotropy group W_x and its Coxeter number h_x .

About the partial derivatives at the origin (or a point x where $W_x = W$), since the $M_{i,j}$ are homogeneous of degree $s_j = \sum_{1 \leq u \leq n, u \neq j} (k_u - 1)$, we see that $(\partial \tilde{F} / \partial p_j) \circ P$ is of class \mathcal{H}^{r-k_j} . Reasoning as above, we could show that the partial derivatives $\partial^{[m]} \tilde{F} / \partial P^m$ of order $m = (m_1, \dots, m_n)$ are continuous if $m_1 k_1 + \dots + m_n k_n \leq r$. For instance the partial derivatives in W -invariant directions are continuous up to the order r .

6. COUNTER EXAMPLE.

Let us give a counter example which applies to almost every finite reflection group. It is sufficient to consider essential irreducible groups.

⁴The closure of each connected component of the regular set being convex and thus Whitney 1-regular ([24]), lemma 4.5 could directly give the result.

⁵The description of W_i given in [3] was correct. Unfortunately in [4] it was not. Although not essential in the reasoning it was misleading. The explicit computations were correct however and gave the best result in the cases of A_n , B_n and $I_2(k)$.

We consider $F : \mathbf{R}^n \rightarrow \mathbf{R}$ defined by $F(y) = y_n^{s+\alpha}$ for some integer s and an $\alpha \in]0, 1[$. F is of class \mathcal{C}^s but not of class \mathcal{C}^{s+1} in any neighborhood of $\{y|y_n = 0\}$. Let P be the Chevalley mapping associated with some finite irreducible Coxeter group W acting on \mathbf{R}^n and consider the composite mapping $F \circ P(x) = p_n^{s+\alpha}(x)$. We study the differentiability of this mapping when $p_n(x) = 0$.

A set of basic invariants is available in [18] for any finite Coxeter group. Disregarding D_n , for any other group there exists an invariant set of linear forms $\{L_1, \dots, L_v\}$ the kernels of which intersect only at the origin, and such that for $i = 1, \dots, n$, $p_i(X) = \sum_{j=1}^v [L_j(X)]^{k_i}$ with k_i s as determined in [7]. With the two exceptions of A_{2n} and $I_2(2p+1)$, k_n is even and therefore $p_n(x)$ vanishes only at the origin. We will not study the two exceptional cases, but a fairly general counter example is given in [1] for symmetric functions and thus for A_n (including $A_2 = I_2(3)$). As usual, D_n does not follow the general line but we may choose $p_n(x) = \sum_1^n x_i^{2(n-1)}$ and the results of the general case apply.

We have $p_n(x) = \sum_1^v [L_i(x)]^{k_n}$, and since $|L_i(x)| \leq a_i|x|$, $i = 1, \dots, v$ for some numerical constants a_i , we have the estimate $|p_n(x)| \leq (\sum_1^v a_i^{k_n})|x|^{k_n} = A|x|^{k_n}$.

Analogously, since $|D^1 L_i(x)| \leq b_i$ for some numerical constants b_i , we get:

$$|D^j p_n(x)| \leq \sum_1^v b_i^j \binom{k_n}{j} |L_i(x)|^{k_n-j} = B_j |x|^{k_n-j}.$$

The derivatives of the composite mapping $p_n^{s+\alpha}(x)$ are given by the Faa di Bruno formula:

$$D^k p_n^{s+\alpha}(x) = \sum_{\mu_1, \dots, \mu_q} \frac{k!}{\mu_1! \dots \mu_q!} D^p y_n^{s+\alpha}(p_n(x)) \left(\frac{D^1 p_n(x)}{1!} \right)^{\mu_1} \dots \left(\frac{D^q p_n(x)}{q!} \right)^{\mu_q},$$

where the sum is over all the q -tuples $(\mu_1, \dots, \mu_q) \in \mathbf{N}^q$ such that $1\mu_1 + \dots + q\mu_q = k$, with $p = \mu_1 + \dots + \mu_q$. There are constants $C_{(\mu_1, \dots, \mu_q)}$ such that:

$$\left| \left(\frac{D^1 p_n(x)}{1!} \right)^{\mu_1} \dots \left(\frac{D^q p_n(x)}{q!} \right)^{\mu_q} \right| \leq C_{(\mu_1, \dots, \mu_q)} |x|^{(k_n-1)\mu_1 + \dots + (k_n-q)\mu_q} = C_{(\mu_1, \dots, \mu_q)} |x|^{k_n p - k},$$

and therefore constants $A_{(\mu_1, \dots, \mu_q)}$ and A such that:

$$|D^k p_n^{s+\alpha}(x)| \leq \sum_{(\mu_1, \dots, \mu_q)} A_{(\mu_1, \dots, \mu_q)} |x|^{k_n(s+\alpha-p)} |x|^{k_n p - k} \leq A |x|^{k_n s + k_n \alpha - k}.$$

This shows that the derivatives of order $k \leq k_n s$ tend to 0 at the origin while the derivatives of order $k_n s + 1$ will not if $\alpha < 1/k_n$. This means that the composite mapping $f = F \circ P$ is of class $\mathcal{C}^{k_n s}$ but not of class $\mathcal{C}^{k_n s+1}$ at $x = 0$ and it factors through F which is of class \mathcal{C}^s and not of class \mathcal{C}^{s+1} . The loss of differentiability is as given in theorem 1.1 and cannot be reduced.

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